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Research Internship

Local–Global Principle

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In this article, we prove the multiple map analogue of the local-global principle [2, Theorem 3.2] for periodic orbits of polynomial maps. Indeed, the proof is essentially the same with the single map case.

1 Notations and Terminologies

Let R be a commutative ring with unity of characteristic zero, and N be a positive integer. For a collection of endomorphisms $S \subseteq \mathrm{End}(R^N)$, generating a monoid $\langle S \rangle$ under composition, we say that a point $x \in R^N$ is S-*periodic* if its S-orbit $\mathcal{O}_S(x) = \{f(x) : f \in \langle S \rangle\}$ is a finite set and $\langle S \rangle$ acts on $\mathcal{O}_S(x)$ by permutations.

We identify a positive integer with the corresponding cardinal number. Let n be a positive integer, κ be a cardinal number, and $\pi : \kappa \to \mathrm{Sym}(n)$ be a set map, where $\mathrm{Sym}(n)$ is the symmetric group over the set n. Note that π naturally defines a left action of the free monoid κ^* with κ generators on n. Suppose that this action is transitive. For a collection $S = \{f_j\}_{j\in\kappa} \subseteq \mathrm{End}(R^N)$ consisting of κ endomorphisms that are not necessarily distinct, we call an S-periodic orbit $\mathcal{O}$ is π-*periodic* if the action of $\langle S \rangle$ on $\mathcal{O}$ is given by π, that is, for $\mathcal{O} = \{x_1, \ldots, x_n\}$, we have $f_j(x_i) = x_{\pi_j(i)}$. We call π the *type* of the periodic orbit.

Denote by $\mathrm{Type}(R, N, \kappa)$ the collection of all possible types of periodic orbit given κ polynomial maps in N variables with coefficients in R. More precisely, an element of $\mathrm{Type}(R, N, \kappa)$ is a tuple $(n, \pi : \kappa \to \mathrm{Sym}(n))$ so that there exists a periodic orbit on R^N of type π. Similarly, denote by $\mathrm{Type}(R, N)$ the collection of all possible types of periodic orbit in N variables with coefficients in R, that is,

$$\mathrm{Type}(R, N) = \bigcup \mathrm{Type}(R, N, \kappa).$$

We will prove the following local-global principle.

Theorem 1.1 (The local-global principle)**.** *Let R be a Dedekind domain. For all positive integer $N \geq 2$ and cardinal number κ, we have*

$$\mathrm{Type}(R, N, \kappa) = \bigcap_{\mathfrak{p}\in\mathrm{Spec}\, R\setminus\{0\}} \mathrm{Type}(R_{\mathfrak{p}}, N, \kappa) = \bigcap_{\mathfrak{p}\in\mathrm{Spec}\, R\setminus\{0\}} \mathrm{Type}(\widehat{R}_{\mathfrak{p}}, N, \kappa)$$

and

$$\mathrm{Type}(R, N) = \bigcap_{\mathfrak{p}\in\mathrm{Spec}\, R\setminus\{0\}} \mathrm{Type}(R_{\mathfrak{p}}, N) = \bigcap_{\mathfrak{p}\in\mathrm{Spec}\, R\setminus\{0\}} \mathrm{Type}(\widehat{R}_{\mathfrak{p}}, N).$$

Note that [2, Theorem 3.2] is the $M = 1$ case of this theorem.

2 Elementary Properties of Periodic Orbits

Lemma 2.1. *Let R be a commutative ring with unity of characteristic zero, and N be a positive integer, and κ be a cardinal number. Let $\mathcal{O} = \{x_1, \ldots, x_n\}$ be a periodic orbit on R^N of type $\pi : \kappa \to \mathrm{Sym}(n)$.*

(a) Let $\sigma \in \mathrm{AGL}_N(R)$ be an invertible affine transformation, that is, $\sigma(x) = Ax + b$ for some $A \in \mathrm{GL}_N(R)$ and $b \in R^N$. Then, $\sigma(\mathcal{O})$ is also a periodic orbit of type π.

(b) There exists a periodic orbit of type π whose coordinates of the points are pairwise distinct.

Proof. (a) Suppose that $\mathcal{O}$ is π-periodic with respect to the collection $S \subseteq \mathrm{End}(R^N)$ of endomorphisms. Then, the set $\sigma(\mathcal{O})$ is π-periodic with respect to

$$S^\sigma = \{f^\sigma = \sigma \circ f \circ \sigma^{-1} : f \in S\}.$$

(b) We can achieve this by applying a suitable $\sigma \in \mathrm{AGL}_N(R)$ to $\mathcal{O}$. See [2, Lemma 4.1(v)]. □

3 Local-Global Principle

Theorem 3.1 (The local-global principle). *Let R be a Dedekind domain. For all positive integer $N \geq 2$ and cardinal number κ, we have*

$$\mathrm{Type}(R, N, \kappa) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{0\}} \mathrm{Type}(R_\mathfrak{p}, N, \kappa) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{0\}} \mathrm{Type}(\widehat{R}_\mathfrak{p}, N, \kappa).$$

Proof. Clearly,

$$\mathrm{Type}(R, N, \kappa) \subseteq \mathrm{Type}(R_\mathfrak{p}, N, \kappa) \subseteq \mathrm{Type}(\widehat{R}_\mathfrak{p}, N, \kappa)$$

for all $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{0\}$.

Suppose that $(n, \pi) \in \mathrm{Type}(\widehat{R}_\mathfrak{p}, N, \kappa)$ for all $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{0\}$. For each $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{0\}$ with $\#(R/\mathfrak{p}) < n$, let $\mathcal{O}_\mathfrak{p} = \{x_{\mathfrak{p},1}, \ldots, x_{\mathfrak{p},n}\}$ be a π-periodic orbit in $\widehat{R}_\mathfrak{p}^N$ with respect to the collection $S_\mathfrak{p} = \{f_{\mathfrak{p},j}\}_{j \in \kappa}$ of polynomial maps over $\widehat{R}_\mathfrak{p}$. By Lemma 2.1, we may assume that $x_{\mathfrak{p},i_1}^{(r_1)} \neq x_{\mathfrak{p},i_2}^{(r_2)}$ whenever $(i_1, r_1) \neq (i_2, r_2)$ for each $\mathfrak{p}$.

For each $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{0\}$, let $\mathrm{ord}_\mathfrak{p} : \widehat{R}_\mathfrak{p} \to \mathbb{Z} \cup \{\infty\}$ be the surjective discrete valuation of $\widehat{R}_\mathfrak{p}$. Pick $M \in R$ so that M satisfies:

(i) For each $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{0\}$ with $\#(R/\mathfrak{p}) < n$, we have

$$\mathrm{ord}_\mathfrak{p}(M) > \mathrm{ord}_\mathfrak{p}\left(\prod_{(i_1,r_1) \neq (i_2,r_2)} (x_{\mathfrak{p},i_1}^{(r_1)} - x_{\mathfrak{p},i_2}^{(r_2)})\right).$$

(ii) For each $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{0\}$ with $\#(R/\mathfrak{p}) \geq n$, we have $\mathrm{ord}_\mathfrak{p}(M) = 0$.

Then we construct a nice approximation of $x_{\mathfrak{p},i}$'s in R. Pick $x_i^{(r)} \in R$, $i = 1, \ldots, n$ and $r = 1, \ldots, N$ satisfying:

(i) For each $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{0\}$ with $\#(R/\mathfrak{p}) < n$, we have

$$\mathrm{ord}_\mathfrak{p}(x_{\mathfrak{p},i}^{(r)} - x_i^{(r)}) \geq n\, \mathrm{ord}_\mathfrak{p}(M).$$

(ii) For each $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{0\}$ with $\#(R/\mathfrak{p}) \geq n$, we have

$$\min\left\{\mathrm{ord}_\mathfrak{p}\left(\prod_{i_1 \neq i_2} (x_{i_1}^{(1)} - x_{i_2}^{(1)})\right), \mathrm{ord}_\mathfrak{p}\left(\prod_{i_1 \neq i_2} (x_{i_1}^{(2)} - x_{i_2}^{(2)})\right)\right\} = 0.$$

First, we pick $\widetilde{x}_1, \ldots, \widetilde{x}_n \in R^N$ satisfying (i). Then, we pick $a_1, \ldots, a_n \in R$ so that the points

$$x_i = (\widetilde{x}_i^{(1)} + a_1 M^n, \widetilde{x}_i^{(2)}, \ldots, \widetilde{x}_i^{(N)})$$

satisfy both (i) and (ii). Note that there are only finitely many primes $\mathfrak{p} \in \operatorname{Spec} R\backslash\{0\}$ with $\#(R/\mathfrak{p}) \geq n$ such that

$$\operatorname{ord}_{\mathfrak{p}}\left(\prod_{i_1 \neq i_2} (x_{i_1}^{(2)} - x_{i_2}^{(2)})\right) > 0.$$

For such $\mathfrak{p}$, since $\#(R/\mathfrak{p}) \geq n$ and $\operatorname{ord}_{\mathfrak{p}}(M) = 0$, we may pick $a_1, \ldots, a_n \pmod{\mathfrak{p}}$ so that $x_1^{(1)}, \ldots, x_n^{(1)} \pmod{\mathfrak{p}}$ are pairwise distinct. Hence, we can always pick suitable $a_1, \ldots, a_n \in R$.

Now, we construct a collection $S = \{f_j\}_{j \in \kappa}$ of polynomial maps over R so that $\mathcal{O} = \{x_1, \ldots, x_n\}$ is π-periodic with respect to S, that is, $f_j(x_i) = x_{\pi_j(i)}$ for each i, j. Let $\widetilde{f}_j$, $j \in \kappa$ be polynomials over R in N variables satisfying $\widetilde{f}_j \equiv f_{\mathfrak{p},j} \pmod{M^n}$ for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{0\}$ with $\#(R/\mathfrak{p}) < n$. Fix $j \in \kappa$ and $r \in \{1, \ldots, N\}$. We will put

$$\begin{aligned} f_j^{(r)}(X_1, \ldots, X_N) &= \widetilde{f}_j^{(r)}(X_1, \ldots, X_N) \\ &\quad + \sum_{k=0}^{n-1} M^{n-k} \left[b_k \prod_{v=1}^{k} (X_1 - x_v^{(1)}) + B_k \prod_{v=1}^{k} (X_2 - x_v^{(2)}) \right] \end{aligned}$$

for suitable $b_k, B_k \in R$ so that

$$\begin{aligned} x_{\pi_j(i)}^{(r)} &= \widetilde{f}_j^{(r)}(x_i) \\ &\quad + \sum_{k=1}^{i} M^{n-k} \left[b_k \prod_{v=1}^{k-1} (x_i^{(1)} - x_v^{(1)}) + B_k \prod_{v=1}^{k-1} (x_i^{(2)} - x_v^{(2)}) \right] \end{aligned} \tag{1}$$

for all $i = 1, \ldots, n$.

We inductively choose coefficients $b_k, B_k \in R$, $k = 1, \ldots, n$. Note that the equation (1) only depends on the coefficients b_k's and B_k's with $k \leq i$. Assume that the coefficients $b_1, \ldots, b_{l-1}$ and $B_1, \ldots, B_{l-1}$ are chosen so that the equation (1) holds for $i = 1, \ldots, l-1$. We will choose b_l and B_l so that (1) also holds for $i = l$, that is,

$$\begin{aligned} x_{\pi_j(l)}^{(r)} &= \widetilde{f}_j^{(r)}(x_l) \\ &\quad + \sum_{k=1}^{l} M^{n-k} \left[b_k \prod_{v=1}^{k-1} (x_l^{(1)} - x_v^{(1)}) + B_k \prod_{v=1}^{k-1} (x_l^{(2)} - x_v^{(2)}) \right]. \end{aligned} \tag{2}$$

The above equation reduces to the linear equation

$$A_1 b_l + A_2 B_l = A$$

where

$$\begin{aligned} A_1 &= M^{n-l} \prod_{v=1}^{l-1} (x_l^{(1)} - x_v^{(1)}), \\ A_2 &= M^{n-l} \prod_{v=1}^{l-1} (x_l^{(2)} - x_v^{(2)}), \\ A &= x_{\pi_j(l)}^{(r)} - \widetilde{f}_j^{(r)}(x_l) \\ &\quad - \sum_{k=1}^{l-1} M^{n-k} \left[b_k \prod_{v=1}^{k-1} (x_l^{(1)} - x_v^{(1)}) + B_k \prod_{v=1}^{k-1} (x_l^{(2)} - x_v^{(2)}) \right]. \end{aligned}$$

Hence, it sufficies to show that

$$\gcd(A_1, A_2) \mid A,$$

or equivalently,

$$\operatorname{ord}_{\mathfrak{p}}(A) \geq \min\{\operatorname{ord}_{\mathfrak{p}}(A_1), \operatorname{ord}_{\mathfrak{p}}(A_2)\}$$

for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{0\}$. If $\#(R/\mathfrak{p}) \geq n$, then by our construction of M and $x_i^{(r)}$'s, we have

$$\begin{aligned}\min\{\operatorname{ord}_{\mathfrak{p}}(A_1), \operatorname{ord}_{\mathfrak{p}}(A_2)\} &= \operatorname{ord}_{\mathfrak{p}}(M^{n-l}) \min\left\{\operatorname{ord}_{\mathfrak{p}}\left(\prod_{v=1}^{k-1}(x_l^{(1)} - x_v^{(1)})\right), \operatorname{ord}_{\mathfrak{p}}\left(\prod_{v=1}^{k-1}(x_l^{(2)} - x_v^{(2)})\right)\right\} \\ &= 0.\end{aligned}$$

If $\#(R/\mathfrak{p}) < n$, then $\operatorname{ord}_{\mathfrak{p}}(A) \geq (n-l+1)\operatorname{ord}_{\mathfrak{p}}(M)$, so we will prove that $\operatorname{ord}_{\mathfrak{p}}(A_s) \leq (n-l+1)\operatorname{ord}_{\mathfrak{p}}(M)$ for $s = 1, 2$, or equivalently,

$$\operatorname{ord}_{\mathfrak{p}}\left(\prod_{v=1}^{k-1}(x_l^{(s)} - x_v^{(s)})\right) \leq \operatorname{ord}_{\mathfrak{p}}(M)$$

for $s = 1, 2$. This holds since

$$x_l^{(s)} - x_v^{(s)} \equiv x_{\mathfrak{p},l}^{(s)} - x_{\mathfrak{p},v}^{(s)} \pmod{\mathfrak{p}^{\operatorname{ord}_{\mathfrak{p}}(M)+1}\widehat{R}_{\mathfrak{p}}},$$

and

$$\operatorname{ord}_{\mathfrak{p}}\left(\prod_{v=1}^{k-1}(x_{\mathfrak{p},l}^{(s)} - x_{\mathfrak{p},v}^{(s)})\right) \leq \operatorname{ord}_{\mathfrak{p}}(M)$$

by construction of M. □

Corollary 3.1.1. *Let R be a Dedekind domain. For all positive integer $N \geq 2$, we have*

$$\operatorname{Type}(R, N) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} R \setminus \{0\}} \operatorname{Type}(R_{\mathfrak{p}}, N) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} R \setminus \{0\}} \operatorname{Type}(\widehat{R}_{\mathfrak{p}}, N).$$

Proof. Clearly,

$$\operatorname{Type}(R, N) \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Spec} R \setminus \{0\}} \operatorname{Type}(R_{\mathfrak{p}}, N) \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Spec} R \setminus \{0\}} \operatorname{Type}(\widehat{R}_{\mathfrak{p}}, N).$$

Suppose that for each $\mathfrak{p} \in \operatorname{Spec} R \setminus \{0\}$, we have $n \in \operatorname{Type}(\widehat{R}_{\mathfrak{p}}, N, \kappa_{\mathfrak{p}})$ for some cardinal number $\kappa_{\mathfrak{p}}$. Let κ be the supremum of all $\kappa_{\mathfrak{p}}$'s. Then we also have $n \in \operatorname{Type}(\widehat{R}_{\mathfrak{p}}, N, \kappa)$ since we may add any number of redundant polynomial maps. Hence, $n \in \operatorname{Type}(R, N, \kappa)$, so $n \in \operatorname{Type}(R, N)$. □

4 Periodic Orbits in Completion

Moreover, when κ is a positive integer, we can show the following result.

Proposition 4.1. *Let R be a discrete valuation ring with a valuation v, and $\widehat{R}$ be its completion with respect to v. Then, for all positive integers N and M, we have*

$$\operatorname{Type}(R, N, M) = \operatorname{Type}(\widehat{R}, N, M).$$

Proof. Clearly, $\mathrm{Type}(R, N, M) \subseteq \mathrm{Type}(\widehat{R}, N, M)$. To show the opposite inclusion, let $\mathcal{O} = \{x_1, \dots, x_n\}$ be a π-periodic orbit in $\widehat{R}^N$ for polynomial maps over $\widehat{R}$ in $S = \{f_1, \dots, f_M\}$. By Lemma 2.1, we may assume that the coordinates of x_i's are pairwise distinct.

Fix $j \in \{1, \dots, M\}$ and $r \in \{1, \dots, N\}$. Put $f_j = (f_j^{(1)}, \dots, f_j^{(N)})$, and write

$$f_j^{(r)}(X_1, \dots, X_N) = c_0 + c_1 X_1 + \cdots + c_{n-1} X_1^{n-1} + g(X_1, \dots, X_N)$$

with $c_k \in \widehat{R}$, $g \in \widehat{R}[X_1, \dots, X_N]$. Note that the numbers c_k, $k = 0, 1, \dots, n-1$ satisfy the system of n linear equations

$$c_0 + c_1 x_i^{(1)} + \cdots + c_{n-1} (x_i^{(1)})^{n-1} = x_{\pi_j(i)}^{(r)} - g(x_i^{(1)}, \dots, x_i^{(N)}), \qquad i = 1, \dots, n.$$

Now, approximate x_i's with points $\widetilde{x}_i$'s in R^N and g with polynomials $\widetilde{g}$ over R sufficiently closely. Do the same for other j's and r's. Then, the similar system of linear equations with approximal coefficients has a solution $\widetilde{c}_k$, $k = 0, 1, \dots, n-1$ in R. Construct polynomials $\widetilde{S} = \{\widetilde{f}_1, \dots, \widetilde{f}_M\}$ over R via $\widetilde{c}_k$'s and $\widetilde{g}$'s, then $\widetilde{O} = \{\widetilde{x}_1, \dots, \widetilde{x}_n\}$ is a π-periodic orbit in R^N with respect to $\widetilde{S}$. $\square$

References

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