Dehn Invariant and Tiling Problems

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Student Directed Seminar

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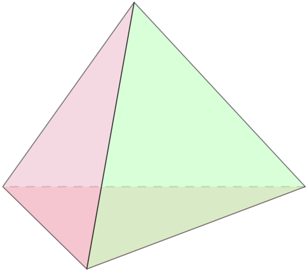
Today's Goals

- *Hilbert's Third Problem*
  부피가 같은 두 다면체가 있을 때, 하나를 유한 조각으로 쪼개어 다른 하나를 만들 수 있는가?
- *Tiling Square with Similar Rectangles*
  닮음인 직사각형들로 정사각형을 채울 수 있는 직사각형의 가로/세로 비율은 무엇인가?
- *Tiling Square with Similar Triangles*
  세 내각이 $30°, 60°, 90°$인 직각삼각형들로 정사각형을 채울 수 있는가?

Overview

Euclidean Geometry



$$V = \frac{1}{3}Sh$$

: The proof involves limiting process. (e.g. Cavalieri's Principle)

$\rightarrow$ Is there a way to define "Area(or, Volume)" elementarily?

Scissors Congruence

Polytopal Decomposition

A *polytopal decomposition* of a n-dimensional polytope P is a finite collections of polytopes $P_1, \ldots, P_n$ whose union is P and which pairwise intersect only in their boundaries.

Scissors Congruence

n-dimensional polytopes P and Q are *scissors congruent* if there exists polytopal decomposition $P_1 \ldots, P_n$ and $Q_1, \ldots, Q_n$ of P and Q, respectively, such that P_i is congruent to Q_i for $1 \leq i \leq n$. We will write $P \sim_{sc} Q$.

Note. Scissors congruence is an equivalence relation.

Scissors Congruence

Motivation. Can we define "Area" via scissors congrunce?

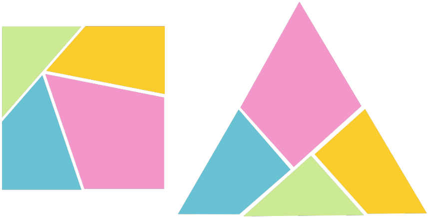
... However, in order to come down to earth again from this height, it is a shame that the equality of the volumes of physical, merely symmetrical, but not congruent structures can only be demonstrated by the exhaustion method and not as elementarily as I know first you showed at the area of the sphirical triangle.

- Carl Friedrich Gauss, April 8, 1844.

In $\mathbb{R}^2$......

Theorem (Wallace-Bolyai-Gerwien; 1807)

Two polygons are scissors congruent iff they have the same area.



Hilbert's Third Problem

Hilbert's Third Problem

Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces that can be reassembled to yield the second?
i.e. Are any two polyhedra of equal volume always scissors congruent?

Remarks.

- It is one of Hilbert's 23 problems, presented in 1900.
- Max Dehn (1878-1952) proved that the answer is "no" by producing a counterexample.

Cauchy's Functional Equation

Cauchy's Functional Equation

Cauchy's functional equation is the functional equation:

$$f(x+y) = f(x) + f(y).$$

Solutions of this equation are called *additive*.

- $f(x) = cx$ is a solution of CFE for $c \in \mathbb{R}$.
- We can easily show that $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$.

Hamel Basis

Axiom (Axiom of Choice)

For any set X of nonempty sets, there exists a choice function f defined on X.

Theorem

Every vector space has a basis.

Corollary (Hamel Basis)

There exists a basis $H = \{h_\alpha\}_{\alpha \in I}$ of $\mathbb{R}$ considered as a $\mathbb{Q}$-vector space, i.e., for all $\beta \in \mathbb{R}$, there exists a unique representation of the form

$$\beta = \sum_{i=1}^{n} q_i h_{\alpha_i}$$

where $q_i \in \mathbb{Q}$ and n depends on β. We call such H a *Hamel basis*.

Nonlinear Additive Solution

Pick a Hamel basis $H = \{h_\alpha\}_{\alpha \in I}$.

For each $x = \sum_{i=1}^{n} q_i h_{\alpha_i}$, we have $f(x) = \sum_{i=1}^{n} q_i f(h_{\alpha_i})$.

$\therefore$ Choose each $f(h_\alpha)$ arbitarily, then the resulting function f is a solution of Cauchy's functional equation.

Note. We may choose some finitely many elements of H arbitarily (if they are linearly independent).

E.g. There exists additive f with $f(1) = 1$, $f(\sqrt{2}) = 0$ and $f(\pi) = -1$.

Dehn Invariant

Dehn Invariant is an invariant preserved under the scissors congruence.

Idea. "Merging the edges" of polyhedra.

The conditions of Dehn invariant we expect:

- Two edges of same length can be "attached side-to-side," so that their angles are added together.
- Two edges of same angle can be "attached end-to-end," so that their lengths are added together.
- An edge of angle π is no edge at all, so it counts as 0.
- It must be nontrivial enough to produce a counterexample.

Dehn Invariant

Dehn Invariant

Let H be a Hamel basis containing π, and f be an additive function induced from H, with $f(\pi) = 0$. For an edge e of a polytope P, denote its length by $\ell(e)$ and its angle by $\theta(e)$. Then, we define the *Dehn invariant* of P by

$$D(P) = \sum_{e\text{: edge}} \ell(e) \cdot f(\theta(e)).$$

Note. There is an equivalent formulation of Dehn invariant:

$$D(P) = \sum_{e\text{: edge}} \ell(e) \otimes (\theta(e) + \pi\mathbb{Q}).$$

We can easily check that it is invariant under scissors congruence.

Disproof of Hilbert's Third Problem (Hadwiger; 1950's)

Lemma

$\arccos \frac{1}{3} \notin \pi\mathbb{Q}$.

Proof.

Denote T_n by the Chebyshev polynomial, defined by $T_n(\cos\theta) = \cos(n\theta)$.

Assume that $\arccos \frac{1}{3} \in \pi\mathbb{Q}$, say $\arccos \frac{1}{3} = \frac{m}{n}\pi$.

Substituting $\theta = \arccos \frac{1}{3}$, we have $T_n(\frac{1}{3}) = \cos(m\pi) = \pm 1$.

From the identity $T_{n+1} = 2xT_n - T_{n-1}$, we can show that the leading coefficient of T_n is 2^{n-1}.

Hence, $\pm 3^n = 3^n T_n(\frac{1}{3}) = 2^{n-1} +$ (multiple of 3), a condtradiction. □

Disproof of Hilbert's Third Problem (Hadwiger; 1950's)

Property (Counterexample of Hilbert's Third Problem)

The cube and the regular tetrahedron of equal volume are not scissors congruent.

Proof.

Let H be a Hamel basis containing π and $\arccos \frac{1}{3}$, and f be an additive function induced from H, with $f(\pi) = 0$ and $f(\arccos \frac{1}{3}) = 1$. Construct a Dehn invariant D using f. Then we have

$$D(\text{cube}) = \sum_e \ell(e) f(\frac{\pi}{2}) = 0 \neq \sum_e \ell(e) f(\arccos \frac{1}{3}) = D(\text{tetrahedron}),$$

hence the cube and the regular tetrahedron are not scissors congruent. □

Note. We may remove AoC from this proof by fixing a polytopal decomposition.

Converse of Dehn Invariant Condition

Q. Then, which polyhedrons are scissors congruent?

Theorem (Sydler; 1965)

Two polyhedron of equal volumes are scissors congruent iff they have the same Dehn invariant.

Notation

- $\mathbb{Q}[x_1, \ldots, x_n]$: The *polynomial ring* on n indeterminates $x_1, \ldots, x_n$ with coefficients in $\mathbb{Q}$.
- $\mathbb{Q}(x_1, \ldots, x_n)$: The *rational function field* on n indeterminates $x_1, \ldots, x_n$ over $\mathbb{Q}$, i.e., the field of fractions of $\mathbb{Q}[x_1, \ldots, x_n]$.

Tiling a Rectangle with Squares

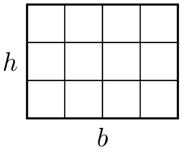
Eccentricity of a Rectangle

For a rectangle R with base b and height h, define the *eccentricity* of R by $r = \frac{h}{b}$.

Theorem (Dehn)

A rectangle can be tiled using finitely many squares if and only if its eccentricity is a rational number.

The 'if' side is trivial, but how can we prove the 'only if' side?



Tiling a Rectangle with Squares

Proof.

Suppose that the eccentricity $r = \frac{h}{b}$ of R is irrational.

Then h and b are linearly independent over $\mathbb{Q}$.

Let H be an additive function with $f(b) = 1$ and $f(h) = -1$, and define the *Hamel area* of any $x \times y$ rectangle by $f(x)f(y)$, then it is additive.

Then the Hamel area of R is $f(b)f(h) = -1 < 0$, but the Hamel area of a square is $f(x)^2 \geq 0$, hence there is no partition of R into squares. □

Note. We may choose $xf(y) - yf(x)$ as the Hamel area function.

Tiling a Rectangle with Rectangles

Theorem (Dehn-Freiling-Rinne)

Let $R_1, \ldots, R_n$ be a rectangular partition of rectangle R_0, and $r_i = \frac{h_i}{b_i}$ be the eccentricity of R_i. Then, $r_0 \in \mathbb{Q}(r_1, \ldots, r_n)$.
Especially, $r_0 = \frac{P(r_1,\ldots,r_n)}{Q(r_1,\ldots,r_n)}$ for some $P, Q \in \mathbb{Q}[r_1, \ldots, r_n]$ where all terms of P are of the same degree, all terms of Q of the same degree, and $\deg P = \deg Q + 1$.

Tiling a Rectangle with Rectangles

Step 1: $r_0 \in \mathbb{Q}(r_1, \ldots, r_n)$

Proof.

Denote the field $F = \mathbb{Q}(r_1, \ldots, r_n) \leq \mathbb{R}$, and consider $\mathbb{R}$ as a F-vector space.

If $r_0 \notin F$, then h_0 and b_0 are linearly independent over F. Let f_1 and f_2 be the additive function which indicates the h_0(and b_0, resp.)-coefficient.

For each $b \times h$ rectangle, define its *Hamel area* by $f_1(b)f_2(h) - f_1(h)f_2(b)$.

Then the Hamel area of R_0 is $f_1(b_0)f_2(h_0) - f_1(h_0)f_2(b_0) = 1 \cdot 1 - 0 \cdot 0 = 1$,

while the Hamel area of R_i is $f_1(b_i)f_2(h_i) - f_1(h_i)f_2(b_i) =$

$f_1(b_i)f_2(r_ib_i) - f_1(r_ib_i)f_2(b_i) = r_i[f_1(b_i)f_2(b_i) - f_1(b_i)f_2(b_i)] = 0$.

Since the Hamel area is additive, that gives a contradiction. $\square$

Tiling a Rectangle with Rectangles

Step 2: $r_0 = \frac{P(r_1,\ldots,r_n)}{Q(r_1,\ldots,r_n)}$ where P, Q have the desired property.

Proof Sketch.

Let $r_0 = \frac{P(r_1,\ldots,r_n)}{Q(r_1,\ldots,r_n)}$, and take η be a trancendental number over F.

Stretch the vertical axis by η times, we have $\eta r_0 = \frac{P_1(\eta r_1,\ldots,\eta r_n)}{Q_1(\eta r_1,\ldots,\eta r_n)}$.

Hence, $\eta P(r_1,\ldots,r_n)Q_1(\eta r_1,\ldots,\eta r_n) = P_1(\eta r_1,\ldots,\eta r_n)Q(r_1,\ldots,r_n)$.

Consider the both sides as polynomials of η, and compare the leading terms. $\square$

Tiling a Square with Similar Rectangles

Corollary

If a square is partitioned by rectangles of eccentricity r, then r is algebraic.

Proof.

$1 = \frac{P(r,1/r)}{Q(r,1/r)}$ where P, Q are the polynomials of the previous theorem.
Since the degree of the terms in P and Q have the opposite parity, this gives a nontrivial polynomial with root r. □

Tiling a Square with Similar Rectangles

Totally Positive

For an algebraic number r, the *conjugate roots* of r are the roots of the minimal polynomial of r. We say r is *totally positive* if all the conjugate roots of r have positive real part.

Theorem (Freiling-Rinne-Laczkovich-Szekeres)

A square can be partitioned using rectangles of eccentricity r iff r is totally positive.

Tiling a Square with Similar Rectangles

($\Leftarrow$):

Theorem (Wall)

Let $P(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_0$ and $Q(x) = p_{n-1}x^{n-1} + p_{n-3}x^{n-3} + \ldots$ be the *alternant* of $P(x)$. All roots of $P(x)$ have positive real part iff

$$\frac{Q(x)}{P(x) - Q(x)} = \frac{-1}{c_n x + \frac{1}{c_{n-1}x + \frac{1}{\cdots + \frac{1}{c_1 x}}}}.$$

Let $P(x)$ be the minimal polynomial of r, then $P(r) = 0$, hence $\frac{Q(r)}{P(r)-Q(r)} = -1$. Put $P(r)$ in the Wall's Theorem, then we can inductively construct the partition of rectangles with eccenticity $\frac{1}{c_k r + \frac{1}{c_{k-1}r + \frac{1}{\cdots + \frac{1}{c_1 r}}}}$. □

Tiling a Square with Similar Rectangles

($\Rightarrow$):

Assume that r is algebraic of degree n while not totally positive.

Denote the minimal polynomial of r by $p(x) \in \mathbb{Q}[x]$.

Lemma

Let $r > 0$ be algebraic with minimal polynomial $p(x)$, then r is totally positive or p has a root with strictly negative real part.

Lemma

If $p(x) \in \mathbb{Q}[x]$ is irriducible, then p has distinct roots.

Hence, $p(x)$ has n distinct roots, and it has a root with strictly negative real part.

Tiling a Square with Similar Rectangles

We construct the *Hamel area* via a quadratic form.

Since $\mathbb{Q}[r]$ forms a field, consider $\mathbb{R}$ as a $\mathbb{Q}[r]$-vector space.

Let H be a Hamel basis of $\mathbb{R}$ containing 1, and denote $p_x(r) = \sum_{i=0}^{n-1} a_i r^i \in \mathbb{Q}[r]$ by the 1-coefficient of x. Write $[x] = (a_0, \ldots, a_{n-1})^T$.

Note that for all symmetric $n \times n$ matrix M, $[b]^T M[h]$ is an additive Hamel area for a $b \times h$ rectangle.

Consider the companion matrix Q of $p(x)$ whose eigenvalues are the roots of $p(x)$. Then we have $[rx] = Q[x]$ for all $x \in \mathbb{R}$.

Tiling a Square with Similar Rectangles

First, assume that all eigenvalues of Q are real. Since all eigenvalues of Q are distinct, we can consider a diagonalization $P^{-1}QP = D$.
Choose $M = (P^{-1})^T DP^{-1}$, then the Hamel area of a rectangle of eccentricity r is $[b]^T M[rb] = [b]^T MQ[b] = [b]^T (P^{-1})^T D^2 P^{-1}[b] = v^T D^2 v \geq 0$.
WLOG, assume that the first entry λ_1 in D is negative.
Then $(Pe_1)^T M(Pe_1) = \lambda_1 < 0$. Hence, take $s \in \mathbb{R}$ with $[s]$ sufficiently close to Pe_1, so that $[s]^T M[s] < \frac{\lambda_1}{2} < 0$.
Since the Hamel area of $s \times s$ square is negative, it cannot be partitioned into rectangles of eccenticity r. Thus, no square can be so partitioned.

The general case can be solved, with the similar argument along the 2×2 block diagonalization. $\square$

Tiling a Square with Similar Triangles

In the problem of tiling a square with right triangles, we get an analogous result.

Theorem (Laczkovich-Szegedy)

A square can be partitioned into right triangle with an acute angle α iff $\tan\alpha$ is totally positive algebraic number.

If we omit the right angle condition, then there are 3 sporadic cases.

Theorem (Laczkovich)

If a square can be partitioned into triangles that are similar with $\triangle$, then $\triangle$ is a right triangle or the angles of $\triangle$ are $(\pi/8, \pi/4, 5\pi/8)$ or $(\pi/4, \pi/3, 5\pi/12)$ or $(\pi/12, \pi/4, 2\pi/3)$.

Tiling a Square with Similar Triangles

We will sketch the proof of the following partial proposition.

Proposition

A square cannot be partitioned into triangles with angles 30°, 60° and 90°.

Tiling a Square with Similar Triangles

Theorem

Suppose that a polygon P is partitioned into triangles $\triangle_1, \ldots, \triangle_N$. Then the coordinates of the vertices of each $\triangle_j$ belong to the field generated by the coordinates of the vertices of P and the cotangents of the angles of the triangles.

Note. Such field contains the slopes of the sides of triangles.

Tiling a Square with Similar Triangles

Theorem

Let P be a polygon with vertices $V_i = (a_i, b_i)$. Suppose that P is partitioned into triangles $\triangle_1, \dots, \triangle_N$ and denote the angles of $\triangle_j$ by $\alpha_j, \beta_j, \gamma_j$. Denote F by the field generated by $a_i, b_i, \cot\alpha_j, \cot\beta_j, \cot\gamma_j$.

If $\phi : F \to \mathbb{R}$ is an isomorphism into its image which leaves a_i, b_i fixed, then there is a j such that at least two of the numbers $\phi(\cot\alpha_j), \phi(\cot\beta_j), \phi(\cot\gamma_j)$ are positive.

Tiling a Square with Similar Triangles

Proof.

Shift the coordinates of $\triangle_j$ via ϕ, and denote the resulting triangle by $\triangle'_j$.

Give the counter-clockwise orientation to each ∂P and $\partial\triangle_j$, and the corresponding orientation to each $\triangle'_j$, then $\partial P = \sum_j \partial\triangle_j = \sum_j \partial\triangle'_j$.

Consider the signed area $A = \int_{\partial P} x dy$ and $A_j = \int_{\partial\triangle'_j} x dy$, then $A = \sum_j A_j$.

Hence, at least one $\triangle'_j$ must be oriented counter-clockwise.

Denote the angles of $\triangle'_j$ by $\alpha'_j, \beta'_j, \gamma'_j$, then with some calculation, we have $\cot\alpha' = \phi(\cot\alpha)$ and analogous equations.

At least two of the angles $\alpha'_j, \beta'_j, \gamma'_j$ are acute, hence at least two of $\phi(\cot\alpha'_j), \phi(\cot\beta_j), \phi(\cot\gamma_j)$ are positive. □

Tiling a Square with Similar Triangles

Proposition

A square cannot be partitioned into triangles with angles 30°, 60° and 90°.

Proof.

Suppose that the unit square has such decomposition.

Since $\cot 30^\circ = \sqrt{3}$ and $\cot 60^\circ = \sqrt{3}/3$, apply the previous theorem on $\mathbb{Q}(\sqrt{3})$.

Let ϕ be the automorphism of $\mathbb{Q}(\sqrt{3})$ with $\phi(\sqrt{3}) = -\sqrt{3}$, then ϕ maps $\cot 30^\circ$ and $\cot 60^\circ$ to negative numbers, a contradiction. $\square$